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DISTRIBUTION MODULO 1 OF SOME OSCILLATING SEQUENCES

BY

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ABSTRACT

Sequences of the form $(P(n)f(Q(n)))_{n=1}^{\infty}$, P and Q polynomials, f a "highly differentiable" periodic function, are considered. The results of [3] concerning the recurrence of this sequence to its value for $n = 0$ are given a quantitative form. Density and uniform distribution modulo 1 are studied for special Q's.

1. Introduction

It is well-known that, given a polynomial P having at least one irrational coefficient (except for the free term), the sequence $(P(n))_{n=1}^{\infty}$ is uniformly distributed modulo 1 (henceforward – u.d.). In particular, the sequence attains values which are arbitrarily close to $P(0)$ modulo 1. Moreover, denoting by $\|t\|$ the distance of a real number t from the nearest integer, there exists a constant $\rho > 0$, depending only on the degree of P, such that $\|P(n) - P(0)\| < 1/n^{\rho}$ for infinitely many positive integers n (see [2, Th. 4.5, Th. 5.2]). For results ensuring simultaneous recurrence of several polynomials see [1, Th. 1] and [6].

A few results are also known for more general functions (cf. [7]). However, some monotonocity assumptions are usually made, and less is known for "oscillating" functions. LeVeque [8] showed that for any increasing sequence of integers $(a_n)_{n=1}^{\infty}$ the sequence $(a_n \cos a_n\alpha)_{n=1}^{\infty})$ is u.d. for almost every α (in the sense of the Lebesgue measure). It is impossible, though, to use his technique to show that this sequence, even in a special case such as, say, $a_n = n$, is u.d. for some specific

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α. Furstenberg and Weiss [6], dealing with the question of small values, noticed that for almost every α the set of positive integer solutions n of

$$\| n \cos n\alpha \| < \epsilon \tag{1.1}$$

contains arbitrarily large "holes" if $\epsilon < 1/2$ (in contrast with the case of polynomials). They asked whether (1.1) has a solution for every α and $\epsilon > 0$.

The last question was answered affirmatively in [3]. More generally, it was proved there that, given real polynomials P and Q and a periodic function f which is sufficiently many times differentiable at $Q(0)$, the inequality

$$\| P(n) f(Q(n)) - P(0) f(Q(0)) \| < \epsilon \tag{1.2}$$

has a positive integer solution n for any $\epsilon > 0$.

Our object of study in this paper is sequences of the form $(P(n)f(Q(n)))_{n=1}^{\infty}$, P and Q polynomials and f periodic. In Section 2 we improve [3, Th. 2.1], showing that the inequality obtained from (1.2) after replacing its right hand side by $1/n^{\rho}$ still has infinitely many solutions for a certain $\rho > 0$, depending only on the degrees of P and Q (we pay for this improvement, though, by requiring f to be slightly more differentiable).

In between the questions of small values and of uniform distribution modulo1 lies the question of density modulo 1. Few classes of sequences are known to be dense modulo 1 without being already u.d. (for an example see [5, Th. IV.1]). In Section 3 we study density modulo 1 for our sequences. The result we obtain is restricted to the case of Q being a monomial, but requires only a minor extra condition on f in addition to what we assume in Section 2 for the small values result (in particular, the assumptions on f still relate only to its behaviour near $Q(0)$).

Section 4 deals with uniform distribution. Here we assume that Q is linear. Under rather mild (but of course global) assumptions on f we are able to show that our sequence is u.d. Our method also provides some bounds on the discrepancy. It is worthwhile to mention that, since our functions are oscillating, the estimation of the arising exponential sums cannot be accomplished in a straightforward fashion employing the methods of van der Corput, Vinogradov or Bombieri–Iwanieć. The main part of our argument is devoted to developing a way of estimating such sums.

2. Small values

Throughout the paper, P will denote a real polynomial of degree d and f a periodic function of period T. Our main result in this section is

THEOREM 2.1. *Let $Q(x) = c_0 + c_1x + \cdots + c_ex^e$. Suppose $(n_k)_{k=1}^{\infty}$ is a sequence of integers such that*

$$\|c_jn_k^j\| < 1/n_k^{\tau}, \qquad 1 \le j \le l, \quad 1 \le k \qquad (\tau > 0).$$

If f is more than d/τ times differentiable at $Q(0)$, then there exists a $\rho > 0$ such that the inequality

$$\|P(n)f(Q(n)) - P(0)f(Q(0))\| < 1/n^{\rho} \tag{2.1}$$

has infinitely many positive integer solutions n.

REMARK 2.1. Going over the proof, it is easy to verify that, if f is only assumed to be d/τ times differentiable at $Q(0)$, then, replacing the right hand side of (2.1) by any $\epsilon > 0$, we still have infinitely many solutions for the resulting inequality.

The theorem, combined with [1, Th. 2], also yields

THEOREM 2.2. *Given non-negative integers d and e, there exist (effective) numbers $s = s(d,e)$ and $\rho = \rho(d,e) > 0$ possessing the following property: For any polynomials P and Q of degrees d and e, respectively, and periodic function f which is s times differentiable at $Q(0)$, (2.1) has infinitely many solutions.*

For $t \in \mathbf{R}$, denote by $\{t\}$ the number in $[-1/2,1/2)$ such that $t - \{t\} \in \mathbf{Z}$.

PROOF OF THEOREM 2.1. We may assume, without loss of generality, that $T = 1$ and $c_0 = 0$. Set $s = [d/\tau] + 1$, and write $f(x) = \sum_{i=0}^{s-1} a_ix^i + O(x^s)$ for appropriate constants a_i, $0 \le i < s$. If $(h_k)_{k=1}^{\infty}$ is any sequence of integers satisfying $h_k = o(n_k^{\tau})$, then for every sufficiently large k we have

$$\begin{aligned} P(h_kn_k)f(Q(h_kn_k)) &= P(h_kn_k)f(\{Q(h_kn_k)\}) \\ &= P(h_kn_k)f(\textstyle\sum_{j=1}^{l} h_k^j\{c_jn_k^j\}) \\ &= P(h_kn_k)\cdot(\textstyle\sum_{i=0}^{s-1} a_i(\sum_{j=1}^{l} h_k^j\{c_jn_k^j\})^i + O(|\sum_{j=1}^{l} h_k^j\{c_jn_k^j\}|^s)) \\ &= P(0)f(Q(0)) + R(h_k) + P(h_kn_k)\cdot O(h_k^{es}/n_k^{\tau s}), \end{aligned} \tag{2.2}$$

where R is a polynomial of degree $d + e(s-1)$ without free term whose coefficients are themselves polynomials in n_k and the $\{c_jn_k^j\}$'s.

We want to define (h_k) in such a way that the last two terms on the right hand side of (2.2) will be as small as possible. Let $(S_k)_{k=1}^{\infty}$ be a sequence of non-negative integers, to be determined subsequently. According to [2, Th. 4.5, Th. 5.2] there exists an effective $\beta > 0$, depending only on $d + e(s-1)$, such that for each k we can find an h_k with $1 \le h_k \le S_k$ satisfying

$$\|R(h_k)\| < 1/S_k^{\beta}. \tag{2.3}$$

For the last term on the right hand side of (2.2) we then have

$$P(h_k n_k)\cdot O(h_k^{es}/n_k^{\tau s}) = S_k^{d+es}\cdot O(n_k^{d-\tau s}). \tag{2.4}$$

Combining (2.2), (2.3) and (2.4) we obtain

$$\|P(h_k n_k) f(Q(h_k n_k)) - P(0) f(Q(0))\| < 1/S_k^\beta + S_k^{d+es}\cdot O(n_k^{d-\tau s}). \tag{2.5}$$

Denote $\gamma = (\tau s - d)/(d + es + \beta) > 0$. Choosing $S_k = [n_k^\gamma]$ we get

$$\begin{aligned}\|P(h_k n_k) f(Q(h_k n_k)) - P(0) f(Q(0))\| &\ll 1/S_k^\beta \ll 1/(S_k n_k)^{\beta\gamma/(\gamma+1)} \\ &\le 1/(h_k n_k)^{\beta\gamma/(\gamma+1)}.\end{aligned}$$

This proves the theorem.

Remark 2.2. If f has more than $[d/\tau] + 1$ derivatives at $Q(0)$, then in the proof we have several choices concerning the s for which we use the representation $f(x) = \sum_{i=0}^{s-1} a_i x^i + O(x^s)$. Enlarging s results in a larger second term and a smaller third term on the right hand side of (2.2). With the results currently available on small values of polynomials, the bound on the second term increases quite rapidly with the degree of R (or, equivalently, with s), so enlarging s does not seem to be worthwhile; the situation may change if this bound is decreased.

Example 2.1. Consider the sequence $(nf(n\alpha))_{n=1}^\infty$. Making no special assumption on the approximation properties of α, we can use Theorem 2.1 with $\tau = 1$. To employ the theorem, we need f to be twice differentiable at 0. With the notations in the proof, we have $\beta = 1/2 - \epsilon_1$ for arbitrary $\epsilon_1 > 0$ [9]. Then $\gamma = 1/(7/2 - \epsilon_1)$, and therefore $\rho = (1/2 - \epsilon_1)/(9/2 - \epsilon_1) - \epsilon_2$ for arbitrary $\epsilon_2 > 0$, namely $\rho = 1/9 - \epsilon$ for arbitrary $\epsilon > 0$. Thus, if f is twice differentiable at 0, then the inequality $\|nf(n\alpha)\| < 1/n^{1/9-\epsilon}$ has infinitely many solutions n for any $\epsilon > 0$. In particular, this is the case with the inequalities $\|n\beta \cos n\alpha\| < 1/n^{1/9-\epsilon}$ and $\|n\beta \sin n\alpha\| < 1/n^{1/9-\epsilon}$. We shall now see that in these special cases we can make some improvements.

Proposition 2.1. *For any real numbers α and β, each of the following inequalities has infinitely many solutions n:*

(i) $\|n\beta \cos n\alpha\| < C/n^{1/5}$ *($C > 0$ a certain absolute constant).*

(ii) $\|n\beta \sin n\alpha\| < 1/n^{2/13-\epsilon}$ *($\epsilon > 0$ arbitrary).*

Proof. Follow the proof of Theorem 2.1. In (i) we take $s = 2$ and gain from the fact that $\cos'(0) = 0$, which makes R linear. Thus, instead of (2.5) we have

$$\|h_k n_k \beta \cos h_k n_k \alpha\| < 1/S_k + S_k^3 \cdot O(1/n_k),$$

and selecting $S_k = [n_k^{1/4}]$ we obtain the required result. For (ii) we pick $s = 3$ and notice that, since $\sin''(0) = 0$, the polynomial R is quadratic only. It follows that

$$\| h_k n_k \beta \sin h_k n_k \alpha \| < 1/S_k^{1/2-\epsilon_1} + S_k^4 \cdot O(1/n_k^2)$$

for arbitrary $\epsilon_1 > 0$. Proceeding as in the proof of Theorem 2.1, we arrive at (ii).

For $\beta = 1$ (actually, for any rational β) one can do much better.

PROPOSITION 2.2. *There exists a constant C such that, for any real α, each of the following inequalities has infinitely many solutions*:

(i) $\| n \cos n\alpha \| < C/n$.

(ii) $\| n \sin n\alpha \| < C/n$.

PROOF. Write $\alpha = 2\pi\alpha_1$.

(i) If $\| n\alpha_1 \| < 1/n$ then

$$\begin{aligned} \| n \cos n\alpha \| &= \| n - n \cos n\alpha \| \leq 2n \sin^2 n\pi\alpha_1 \\ &= 2n \sin^2 \pi\{ n\alpha_1 \} \leq 2n\pi^2\{ n\alpha_1 \}^2 < 2\pi^2/n. \end{aligned}$$

(ii) If α_1 is rational, then our claim is trivial. Otherwise, there exist infinitely many n's satisfying $\| n\alpha_1 - 1/4 \| < 1/n$ (cf. [4]), and for each such n

$$\| n \sin n\alpha \| = \| n \cos 2\pi\{ n\alpha_1 - 1/4 \} \| \leq 2n \sin^2 \pi\{ n\alpha_1 - 1/4 \} < 2\pi^2/n.$$

This proves the proposition.

The proposition provides the best possible result, at least as far as the power of n on the right hand side of the inequality goes. For example, we have

PROPOSITION 2.3. *For almost every real number α (with respect to the Lebesgue measure) the inequality*:

(i) $\| n \cos n\alpha \| < 1/(n \log^2 n)$ *has infinitely many solutions n*,

(ii) $\| n \cos n\alpha \| < 1/(n \log^{2+\epsilon} n)$ *has only finitely many solutions for any $\epsilon > 0$*.

PROOF. It suffices to consider α's in $[0,2\pi)$.

(i) For almost every α there are infinitely many n's such that $\| n(\alpha/2\pi) \| < 1/(2\pi^2 n \log n)$ (cf. [10]), and for these n's the inequality in question holds.

(ii) Denote by μ the Lebesgue measure. It is easy to check that

$$\mu(\{ \alpha \in [0,2\pi) : \| n \cos n\alpha \| < 1/n \log^{2+\epsilon} n)\}) \leq C/(n \log^{1+\epsilon/2} n)$$

for some constant C. Consequently

$$\sum_{n=1}^{\infty} \mu(\{\alpha \in [0,2\pi) : \| n \cos n\alpha \| < 1/(n \log^{2+\epsilon} n)\}) < \infty,$$

and the Borel–Cantelli lemma yields the desired conclusion.

3. Density

In this section we prove

THEOREM 3.1. *Let P be non-constant, e a positive integer, α a real number with α/T irrational and (n_k) a sequence of integers satisfying*

$$\| n_k^e(\alpha/T) \| = o(1/n_k^\tau) \qquad (\tau > 0). \tag{3.1}$$

If $f^{(s)}(0)$ exists and is non-zero for some $s \geq d/\tau$, then the sequence $(P(n)f(n^e\alpha/T))_{n=1}^{\infty}$ is dense modulo 1.

Similarly to the preceding section, this implies

THEOREM 3.2. *Given positive integers d and e, there exists an (effective) $r = r(d,e)$ having the following property: For any P of degree d, f with $f^{(s)}(0) \neq 0$ for some $s \geq r$ and real number α with α/T irrational, the sequence $(P(n)f(n^e\alpha/T))_{n=1}^{\infty}$ is dense modulo 1.*

Another straightforward consequence is

COROLLARY 3.1. *If P is non-constant, f analytic and non-constant and α/T irrational, then $(P(n)f(n^e\alpha/T))_{n=1}^{\infty}$ is dense modulo 1.*

PROOF OF THEOREM 3.1. We may again assume that $T = 1$. Write $f(x) = \sum_{i=0}^{s} a_i x^i + o(x^s)$ for suitable a_i's. If $(h_k)_{k=1}^{\infty}$ is a sequence of integers satisfying

$$h_k = o(\| n_k^e \alpha \|^{-1/e}),$$

then, similarly to (2.2),

$$P(h_k n_k) f((h_k n_k)^e \alpha) = R(h_k) + P(h_k n_k) \cdot o(h_k^{es} \| n_k^e \alpha \|^s) \tag{3.2}$$

for every sufficiently large k, R being a polynomial of degree $d + es$ whose coefficients are themselves polynomials in n_k and $\{n_k^e \alpha\}$. We may assume P to be monic, so that the leading coefficient of R is $a_s n_k^d \{n_k^e \alpha\}^s$. Now by (3.1)

$$| a_s n_k^d \{ n_k^e \alpha \}^s | = o(n_k^{d - \tau s}) = o(1). \tag{3.3}$$

The numbers $a_s n_k^d \{n_k^e \alpha\}^s$, $k \geq 1$, are non-zero. Replacing (n_k) by a subsequence thereof, we may assume all of them to be of the same sign, say positive. Pick an arbitrary irrational $\theta > 0$. For an arbitrary fixed positive integer h, define the sequence (h_k) by $h_k = hr_k$, where

$$r_k = [(\theta/(a_s n_k^d \{n_k^e \alpha\}^s))^{1/(d+es)}]. \tag{3.4}$$

It is readily verified that (3.2) is satisfied. For the second term on the right hand side of (3.3) we have

$$P(h_k n_k) \cdot o(h_k^{es} \| n_k^e \alpha \|^s) \ll (h_k n_k)^d \cdot o(h_k^{es} \| n_k^e \alpha \|^s) = o(1).$$

Now write $R(h_k) = R_1(h)$ where R_1 is a polynomial of degree $d + es$ whose coefficients are polynomials in r_k, n_k and $\{n_k^e \alpha\}$. In view of (3.4), $r_k \xrightarrow[k\to\infty]{} \infty$. The leading coefficient of R_1, namely $a_s r_k^{d+es} n_k^d \{n_k^e \alpha\}^s$, converges therefore to θ as $k \to \infty$. Passing to subsequences, we may assume all other coefficients of R_1 to converge modulo 1 as $k \to \infty$. Consequently

$$P(hr_k n_k) f((hr_k n_k)^e \alpha) \xrightarrow[k\to\infty]{} R_2(h) \text{ (mod. 1)},$$

R_2 being a polynomial of degree $d + es$ whose leading coefficient is irrational. Thus $\{R_2(h) : h \in \mathbf{N}\}$ is dense modulo 1, and thereby the theorem is proved.

EXAMPLE 3.1. Suppose $T = 1$. If $f^{(s)}(0)$ exists and is non-zero for some $s \geq 2$, then the sequence $(nf(n\alpha))_{n=1}^{\infty}$ is dense modulo 1 for any irrational α. Restricting α to the set of non-badly approximable numbers, it suffices to assume the above for some $s \geq 1$ to reach the same conclusion.

EXAMPLE 3.2. The sequence $(n\beta \cos n\alpha)_{n=1}^{\infty}$ is dense modulo 1 unless α and β satisfy one of the following conditions:

(i) $\beta = 0$.

(ii) α is a multiple of $\pi/2$ and β rational.

(iii) α is a multiple of $\pi/3$ and β rational.

Obviously, in each of these three cases the sequence is finite modulo 1. Assume now that $\beta \neq 0$ and write $\alpha = 2\pi\alpha_1$. If α_1 is irrational then our sequence is dense modulo 1 by Theorem 3.1. Suppose therefore that $\alpha = p/q$ with $(p, q) = 1$. If $\cos \alpha$ is irrational then our sequence is clearly dense modulo 1. Thus we may assume that $\cos \alpha$ is rational, and therefore $e^{i\alpha}$ is an algebraic number of degree not exceeding 2 over $\mathbf{Q}$. Now $e^{i\alpha}$ is also a primitive root of unity of order q, so that $\varphi(q) \leq 2$, φ being the Euler totient function. This implies that q is one of the numbers 1,2,3,4,6. Going over the finitely many possibilities we obtain our claim.

4. Uniform distribution and discrepancy

Our basic result in this section is

THEOREM 4.1. *Let P be non-constant. Assume that the set F of points of $[0,T]$, at which either f is not $2d+1$ times continuously differentiable or $f^{(j)}$ vanishes for some $1 \le j \le 2d$, is finite and that $f^{(2d)}$ is bounded. Suppose that α/T is irrational. Then the sequence $(P(n)f(n\alpha/T))_{n=1}^{\infty}$ is u.d.*

REMARK 4.1. It is easy to see that the conclusion of the theorem still holds if the set F is infinite but "sufficiently small", for example, if it has only finitely many accumulation points; also, the requirement that $f^{(2d)}$ be bounded is superfluous. The extra assumptions were introduced with an eye towards Theorem 4.2.

To state our second result we need

DEFINITION 4.1. The discrepancy of $x_1, x_2, \dots, x_N \in [-1/2, 1/2)$ is

$$D((x_i)_{i=1}^{N}) = \sup \left| \frac{1}{N} \sum_{i=1}^{N} \chi_I(x_i) - |I| \right|,$$

the supremum being taken over all intervals $I = [a,b) \subseteq [-1/2,1/2)$ and χ_I denoting the characteristic function of I.

THEOREM 4.2. *Let P, f, F and α be as in Theorem 4.1. Put*

$$a(y) = \min_{1 \le j \le 2d} \min_{x: \rho(x,F) \ge y} |f^{(j)}(x)|,$$

where $\rho(x,F)$ is the distance from x to F. For any N, let $b(N) = b(\alpha,N)$ denote the largest integer $\le N^{\rho}$ with $\|b(N)\alpha/T\| \le N^{-\rho}$, $\rho = (2d+3)/(4d+4)$. Assume that $\epsilon(N)$ is a decreasing function vanishing at infinity and satisfying the following conditions for sufficiently large N:

(1) $\epsilon(N)^{16 \cdot 8^d + 8d} a(\epsilon(N))^{80d+64} \ge 1/N$.

(2) $\epsilon(N)^{8^d+1} a(\epsilon(N))^d \ge 1/b(N)$.

(3) $N\epsilon(N)$ is increasing.

Then $D(N) \equiv D((\{P(n)f(n\alpha/T)\})_{n=1}^{N}) \ll \epsilon(N)\log^2\epsilon(N)$, where the implied constant depends only on P and f.

We shall first prove Theorem 4.1. The proof will be carried out in such a way that it will contain all the ingredients required for the proof of Theorem 4.2.

Throughout this section j is a positive integer and $J = 2^j$. The following lemma is a version of van der Corput's estimate. For $j = 2$ it appears here in its usual form (cf. [11, Th. 5.13]) and for general j can be easily proved by induction. Denote $e(x) = e^{2\pi i x}$. In Lemmas 4.1–4.3 we take $X = X_2 - X_1$.

LEMMA 4.1. *Let $g \in C^j[X_1, X_2]$, and suppose that $0 < \lambda_j \leq g^{(j)}(x) \ll \lambda_j$. Then*

$$\left| \sum_{X_1 \leq x \leq X_2} e(g(x)) \right| \ll X\lambda_j^{1/(J-2)} + 1 + X^{1-2/J} + X(\lambda_j X^{4-8/J})^{-2/J},$$

the implied constants depending only on j.

The following lemma is used instead of the preceding one in case λ_2 is "small".

LEMMA 4.2. *Let $g \in C^2[X_1, X_2]$. Suppose that $0 < \lambda_2 \leq g''(x) \leq C\lambda_2$ and $\max_{1 \leq i \leq 2} \|g'(X_i)\| \geq CX\lambda_2$. Then*

$$S = \left| \sum_{X_1 \leq x \leq X_2} e(g(x)) \right| \ll X\sqrt{\lambda_2} + 1 + \min\{X, 1/\sqrt{\lambda_2}, 1/\|g'(X_1)\| + 1/\|g'(X_2)\|\},$$

where the implied constant depends only on C.

PROOF. For $X\lambda_2 \gg 1$ this follows from the case $j = 2$ of Lemma 4.1. If $X\lambda_2 \ll 1$, then, employing the Poisson summation formula, we get

$$S \ll \left| \sum_{g'(X_1) - 1 \leq l \leq g'(X_2) + 1} \int_{X_1}^{X_2} e(g(x) - lx)\, dx \right| + 1.$$

Since $g'(X_2) - g'(X_1) = g''(\xi)X \ll 1$, the last sum consists of $O(1)$ terms. Now, for any l the function $g'(x) - l$ retains its sign over the interval $[X_1, X_2]$, as otherwise we would have

$$g'(X_1) < l < g'(X_2) = g'(X_1) + g''(\xi)X,$$

which implies $\max_{1 \leq i \leq 2} \|g'(X_i)\| < CX\lambda_2$, contradicting the assumptions of the lemma. The integral mean-value theorem now yields for each l

$$\left| \int_{X_1}^{X_2} e(g(x) - lx)\, dx \right| = \frac{1}{2\pi} \left| \int_{X_1}^{X_2} \frac{1}{g'(x) - l}\, de(g(x) - lx) \right|$$

$$\leq \frac{1}{\|g'(X_1)\|} + \frac{1}{\|g'(X_2)\|}.$$

This proves the lemma.

LEMMA 4.3. *Let $g \in C^j[X_1, X_2]$ and $S = \left|\sum_{X_1 \leq x \leq X_2} e(g(x))\right|$. Then, for any $q \leq X^{2/J}$,*

$$S^J \ll 1 + X^J/q^{J/2} + (X/q)^{J-1} \sum_{h_1=1}^{a} \sum_{h_2=1}^{q^2} \cdots \sum_{h_j=1}^{q^{J/2}} \left| \sum_{X_1 \le x \le X_2 - h_1 - \cdots - h_j} e(g_1(x)) \right|,$$

where

$$g_1(x) = h_1 \cdots h_j \int_0^1 \cdots \int_0^1 g^{(j)}(x + t_1 h_1 + \cdots + t_j h_j)\, dt_j \cdots dt_1.$$

In fact, for $j = 1$ this is the well-known Weyl–van der Corput inequality, and the general case follows by induction.

Lemma 4.4. *Let $g(x) = \sum_{k=1}^{K} A_k x^{\alpha_k} + \sum_{l=1}^{L} B_l x^{-\beta_l}$ with A_k, α_k, B_l, $\beta_l > 0$. Then, for $B \ge A > 0$,*

$$\min_{A \le x \le B} g(x) \ll \sum_{k=1}^{K} A_k A^{\alpha_k} + \sum_{l=1}^{L} B_l B^{-\beta_l} + \sum_{k=1}^{K} \sum_{l=1}^{L} (A_k^{\beta_l} B_l^{\alpha_k})^{1/(\alpha_k + \beta_l)},$$

where the implied constant depends on the α_k's and β_l's only.

For $K = L = 1$ the inequality is routinely proved, and we again continue by induction.

Lemma 4.5. *Let f and $a(y)$ be as in Theorem 4.2 and*

$$g_k(r) = \sum_{i=0}^{d} \binom{j+k}{i} \frac{1}{(d-i)!} t^i f^{(j-i+k)}(x_0 + r/q),$$

$$k = 0,1 \quad (t \ge \epsilon > 0, j \ge d).$$

Then the number of integral solutions $r \in [0, q)$ of the system

$$|g_k(r)| \le a(\delta)(1 + t^d)\epsilon^{2d}\delta^d/6^{d+1}, \qquad k = 0,1$$

is $\ll q\delta + 1$, the implied constant depending only on f and j.

Proof. Use induction on d. For $d = 0$, our system may be written as

$$|f^{(j+k)}(x_0 + r/q)| \le a(\delta)/6, \qquad k = 0,1.$$

For any such r we have $\rho(x_0 + r/q, F) \le \delta$, and the number of r's satisfying the latter inequality is clearly $\ll q\delta + 1$.

Now consider a general $d \ge 1$, and assume the lemma to be valid for $d - 1$. The number of those solutions of our system, satisfying in addition the inequality

$$|g_0'(r)| \ge a(\delta)(1 + t^d)\epsilon^{2d}\delta^d/(6^{d+1}q\delta),$$

is $\ll q\delta + 1$. For the remaining r's we have

$$\left| \sum_{i=0}^{d} \binom{j}{i} \frac{1}{(d-i)!} t^i f^{(j+1-i)}(x_0 + r/q) \right| \leq a(\delta)(1 + t^d)\delta^{d-1}\epsilon^{2d}/6^{d+1}.$$

Using also the original inequality for $g_1(r)$ we get

$$\left| \sum_{i=0}^{d} \left[\binom{j+1}{i} - \binom{j}{i} \right] \frac{1}{(d-i)!} t^i f^{(j+1-i)}(x_0 + r/q) \right|$$
$$\leq 2a(\delta)(1 + t^d)\delta^{d-1}\epsilon^{2d}/6^{d+1},$$

which, after simplification, takes the form

$$\left| \sum_{i=0}^{d-1} \binom{j}{i} \frac{1}{(d-1-i)!} t^i f^{(j-i)}(x_0 + r/q) \right| \leq 2a(\delta)(1 + t^d)\delta^{d-1}\epsilon^{2d}/(6^{d+1}t). \tag{4.1}$$

Combined with our inequality for $g_0(r)$, multiplied by d, this yields

$$\left| \sum_{i=0}^{d} \binom{j}{i} \frac{i}{(d-i)!} t^i f^{(j-i)}(x_0 + r/q) \right| \leq (d + 2/t)a(\delta)(1 + t^d)\delta^{d-1}\epsilon^{2d}/6^{d+1},$$

which can be transformed to

$$\left| \sum_{i=0}^{d} \binom{j-1}{i} \frac{1}{(d-1-i)!} t^i f^{(j-1-i)}(x_0 + r/q) \right|$$
$$\leq (d + 2/t)a(\delta)(1 + t^d)\delta^{d-1}\epsilon^{2d}/(6^{d+1}jt).$$

Taken together with (4.1), this gives

$$\left| \sum_{i=0}^{d-1} \binom{j-1+k}{i} \frac{1}{(d-1-i)!} t^i f^{(j-1-i+k)}(x_0 + r/q) \right|$$
$$\leq a(\delta)(1 + t^{d-1})\epsilon^{2d-2}\delta^{d-1}/6^d, \qquad k = 0, 1.$$

Using the induction hypothesis, this completes the proof.

PROOF OF THEOREM 4.1. Without loss of generality (both here and in the proof of Theorem 4.2) we may assume that P is monic and $T = 1$. For an arbitrary fixed positive integer l, put

$$S = \left| \sum_{N/2 < n \leq N} e(lP(n)f(n\alpha)) \right|.$$

In view of Weyl's criterion for equidistribution we just have to show that $S = o(N)$.

Let $\epsilon_0 > 0$ be arbitrary. Put $\epsilon_1 = \epsilon_0^{8^d} a(\epsilon_0)^d$ and $\epsilon_2 = a(\epsilon_0)$, where $a(y)$ is as in Theorem 4.2. Let N be an integer satisfying

(i) $1/N \ll l^{-8d} \epsilon_1^{16} \epsilon_2^{64d+64}$,

(ii) $l/b(N) \ll \epsilon_1$,

where the implied constants depend only on P and f. We can then write

$$\alpha = p/q + \theta/(qN^\rho) = p/q + 1/Q \qquad (1/b(N) \le q \le N^\rho, (p,q) = 1, |\theta| \le 1).$$

Assume for the sake of convenience that $Q > 0$. Write $P(x) = \sum_{i=0}^d b_i x^{d-i}$. Let us distinguish between two cases:

Case I: $Q \ge cN/\epsilon_2$, where $c = 4^d \max_{1 \le j \le 2d} \sup_x |f^{(j)}(x)| \sum_{i=1}^d |b_i|$.

For a typical $n \in (N/2, N]$ write $n = mq + k$ with $0 \le k < q$, and define r by $r \equiv kp$ (mod. q) and $0 \le r < q$. Since

$$n\alpha = np/q + n/Q \equiv kp/q + n/Q \equiv r/q + n/Q \text{ (mod. 1)},$$

we have

$$lP(n)f(n\alpha) = lP(n)f(r/q + n/Q).$$

For a fixed k, denote the right hand side of this formula by $g(m)$. The number of k's for which $\rho(r/q, F) \le \epsilon_0$ is $\ll \epsilon_0 q$. For any other k

$$\begin{aligned} \left| \frac{\partial^j}{\partial m^j} f(r/q + n/Q) \right| &= |(q/Q)^j f^{(j)}(r/q + n/Q)| \\ &\ge (1 - 4^{-d}) \epsilon_2 (q/Q)^j, \qquad 1 \le j \le 2d. \end{aligned}$$

Denote by $\sum_k^*$ summation over all such k's. If $d \le j \le 2d + 1$, then

$$|g^{(j)}(m)| = \left| lq^j \sum_{i=0}^d \binom{j}{i} P^{(i)}(n) f^{(j-i)}(r/q + n/Q) Q^{j-i} \right| \gg \epsilon_2 l q^j Q^{d-j} \tag{4.2}$$

and

$$|g^{(j)}(m)| \ll l q^j Q^{d-j}. \tag{4.3}$$

Let $u \ge 1$ be the least integer with $lq^{d+u}/Q^u \le \epsilon_3 = \epsilon_0^{2 \cdot 4^d} \epsilon_2^d$.

If $\epsilon_3 \epsilon_0^{J/2} l q^{d+u} Q^{-u} \ge (q/N)^{4-8/J}$, then, applying Lemma 4.1 with $j = d + u$, we obtain

$$
\begin{aligned}
S &= |\textstyle\sum_k \sum_m e(g(m))| \\
&\le N\epsilon_0 + \textstyle\sum_k^* |\sum_m e(g(m))| \\
&\ll N\epsilon_0 + \textstyle\sum_k^* [(N/q)(lq^{d+u}Q^{-u})^{1/(J-2)} + 1 + (N/q)^{1-2/J} \\
&\qquad + (N/q)((Q^u/lq^{d+u}\epsilon_3)(q/N)^{4-8/J})^{2/J}] \\
&\ll N\epsilon_0.
\end{aligned} \tag{4.4}
$$

Suppose therefore that $\epsilon_3\epsilon_0^{J/2}lq^{d+u}/Q^u \le (q/N)^{4-8/J}$. If $j > 2$, then, employing Lemma 4.3 with $j_1 = j - 2$, $J_1 = 2^{j_1}$ and $q_1 = \epsilon_0^{-2}$, we get

$$
\begin{aligned}
S^{J_1} &\ll (N\epsilon_0)^{J_1} + (\textstyle\sum_k^* |\sum_{M_1<m\le M_2} e(g(m))|)^{J_1} \\
&\le (N\epsilon_0)^{J_1} + q^{J_1-1}\textstyle\sum_k^* |\sum_{M_1<m\le M_2} e(g(m))|^{J_1} \\
&\ll (N\epsilon_0)^{J_1} + q^{J_1-1}\textstyle\sum_k^* [1 + (N/q\sqrt{q_1})^{J_1} \\
&\qquad + (N/qq_1)^{J_1-1}\textstyle\sum_{h_1=1}^{q_1}\cdots\sum_{h_{j_1}=1}^{q_1^{J_1/2}} |\sum_m e(g_1(m))|],
\end{aligned} \tag{4.5}
$$

where

$$
g_1(m) = H\int_0^1 \cdots \int_0^1 g^{(j_1)}(m + t_1h_1 + \cdots + t_{j_1}h_{j_1})\, dt_{j_1}\cdots dt_1,
$$

$$
H = h_1h_2\cdots h_{j_1}.
$$

Obviously, (4.5) holds also if $j = 2$. Our present aim is to apply Lemma 4.2 to the sum over m on the right hand side of (4.5). To this end we need to perform some preliminary calculations.

$$
\lambda_2 \sim g_1''(m) = H\int_0^1 \cdots \int_0^1 g^{(j)}(m + t_1h_1 + \cdots + t_{j_1}h_{j_1})\, dt_{j_1}\cdots dt_1.
$$

In view of (4.2) and (4.3)

$$
\epsilon_2 \ll \lambda_2/(Hlq^jQ^{d-j}) \ll 1.
$$

Also

$$
g_1'(m) = H\int_0^1 \cdots \int_0^1 g^{(j-1)}(m + t_1h_1 + \cdots + t_{j_1}h_{j_1})\, dt_{j_1}\cdots dt_1.
$$

For fixed $h_1, h_2, \ldots, h_{j_1}$ and M we now want to estimate the number of k's satisfying $\|g_1'(M)\| \ll \delta$, where $\delta \gg Hlq^jQ^{-u}\cdot N/q$ is a parameter to be determined later. According to (4.2), this is equivalent to

$$\|g_1'(M)\| = \left\| H\binom{j-1}{d} d!\, lq^{j-1} f^{(u-1)}(r/q) Q^{1-u} \right\| + O(Hlq^{j-1}NQ^{-u}) \ll \delta,$$

whence also to

$$\left\| H\binom{j-1}{d} d!\, lq^{j-1} f^{(u-1)}(r/q) Q^{1-u} \right\| \ll \delta. \tag{4.6}$$

Denote by $\sum_k^{**}$ summation over all k's for which (4.6) holds. Put

$$A = \binom{j-1}{d} Q^{1-u} H d!\, lq^{j-1}.$$

Employing a well-known result of Vinogradov (cf. [2, Lemma 2.1]) we obtain

$$\textstyle\sum_k^{**} 1 \ll \min\limits_{\Delta \ge \delta} \left\{ q\Delta + \sum\limits_v \min\{\Delta, 1/(\Delta v^2)\} \left| \sum\limits_r e(vAf^{(u-1)}(r/q)) \right| \right\}.$$

Dividing the interval $[0,1]$ into subintervals over which $f^{(i)}$ $(i = 0, 1, \dots, 2d+1)$ are monotone, we take r such that r/q lies in one of them.

Since $lHq^{2u-3} \le Q^{u-1}\epsilon_3$, we can find the smallest integer $j_2 \le d + 2 - u$ satisfying $lHq^{d+u-1-j_3}Q^{1-u} \le \epsilon_3$. Applying Lemma 4.1 with $j = j_2$ to the last sum over r and later Lemma 4.4 to select an optimal Δ, we get

$$\begin{aligned}
\textstyle\sum_k^{**} 1 &\ll \min\nolimits_{\Delta \ge \delta}\{q\Delta + \textstyle\sum_v \min\{\Delta, 1/\Delta v^2\}\,[q(v\epsilon_3)^{1/(J_2-2)} + q^{1-2/J_2} + q \cdot (qv\epsilon_3)^{-2/J_2}]\} \\
&\ll \min\nolimits_{\Delta \ge \delta}\{q\Delta + q(\epsilon_3/\Delta)^{1/(J_2-2)} + q^{1-2/J_2} + q(\Delta/q\epsilon_3)^{2/J_2}\} \qquad (4.7) \\
&\ll q\delta + q^{1-2/3J_2} + q\epsilon_3^{1/J_2} + q(\delta/q\epsilon_3)^{2/J_2}.
\end{aligned}$$

Using Lemma 4.2, and taking into account (4.4) and (4.5), we arrive at

$$S^{J_1} \ll (N\epsilon_0)^{J_1} + (N/\sqrt{q_1})^{J_1} + (N/q_1)^{J_1-1} \sum_{h_1, \dots, h_{j_1}} \left(\textstyle\sum_k^{**} N/q + q/\delta\right). \tag{4.8}$$

Using (4.7)–(4.8), and Lemma 4.4, applied this time to the parameter δ, we obtain

$$\begin{aligned}
S^{J_1} &\ll (N\epsilon_0)^{J_1} + (N/q)^{J_1-1}\textstyle\sum_{h_1, \dots, h_{j_1}} (N\delta + N(\delta/q\epsilon_3)^{2/J_2} \\
&\qquad\qquad + q/\delta + N\epsilon_3^{1/J_2} + Nq^{-2/3J_2}) \\
&\ll (N\epsilon_0)^{J_1} + N^{J_1}[\sqrt{q/N} + (N\epsilon_3)^{-1/(J_2+2)} + \epsilon_0^{-J} q/N + (\epsilon_0^J \epsilon_3 N)^{-2/J_2}] \\
&\ll (N\epsilon_0)^{J_1}.
\end{aligned}$$

Case II: $Q < cN/\epsilon_2$.
In this case $q \le cN/\epsilon_2$. Set $\epsilon_4 = \epsilon_0^{3d}\epsilon_2^{2d+1}$ and $H = \epsilon_2^2 \epsilon_4 Q$. Then

$$S \le \frac{1}{H} \sum_{n \le N} \left| \sum_{H \le h \le 2H} e(lP(n+h)f(n\alpha + h\alpha)) \right| + N\epsilon_0.$$

Write $h = mq + k$ with $0 \le k < q$ (and $m \sim H/q$). Similarly to the former case, for fixed k and n we denote

$$g(m) = lP(n+h)f(n\alpha + h\alpha) = lP(n+h)f(n\alpha + r/q + h/Q),$$

and obtain

$$g^{(j)}(m) = lq^j \textstyle\sum_{i=0}^{d} \binom{j}{i} P^{(i)}(n+h) f^{(j-i)}(n\alpha + r/q + h/Q) Q^{i-j}, \qquad d \le j \le 2d+1, \tag{4.9}$$

and

$$S \ll N\epsilon_0 + \frac{1}{H} \sum_{n,k} \left| \sum_m e(g(m)) \right|. \tag{4.10}$$

If $Q \ll N\epsilon_0$, then the dominating term in (4.9) is the one with $i = 0$, whence

$$g^{(j)}(m) = lq^j Q^{-j}(n+h)^d f^{(j)}(n\alpha + r/q + h/Q) + O(q^j Q^{1-j} N^{d-1}).$$

The part of the sum on the right hand side of (4.10) corresponding to (n,k)'s such that $\rho(n\alpha + r/q, F) \ll \epsilon_0$ is $\ll (1/H)Nq\epsilon_0(H/q) = N\epsilon_0$. Denote by $\sum_{n,k}^{*}$ summation over the remaining (n,k)'s. For such pairs we have $\epsilon_2 \le |g^{(j)}(m)q^{-j}Q^jN^{-d}/l| \ll 1$. Let $u \ge 1$ be the smallest integer with $l(q/Q)^{d+u}N^d \le \epsilon_0^{2^{d+u}}\epsilon_2^d$. Applying Lemma 4.1 with $j = d + u$ we get

$$\begin{aligned}\frac{1}{H} \textstyle\sum_{n,k}^{*} |\sum_m e(g(m))| &\ll (Nq/H)\,[(H/q)\lambda_j^{1/(J-2)} + (H/q)^{1-1/2J} \\ &\qquad + (H/q)(\lambda_j H^2/q^2)^{-2/J}] \\ &\ll N[\lambda_j^{1/(J-2)} + (q/H)^{1/2J} + (\lambda_j H^2/q^2)^{-2/J}],\end{aligned}$$

where we note that $\epsilon_2 \ll \lambda_j/(lq^jN^dQ^{-j}) \ll 1$. From this we easily conclude that

$$\frac{1}{H} \textstyle\sum_{n,k}^{*} \left| \sum_m e(g(m)) \right| \ll N\epsilon_0,$$

whence $S \ll N\epsilon_0$.

Finally, we need to consider the subcase $N\epsilon_0 \ll Q \ll cN/\epsilon_2$. From (4.9) it follows that

$$g^{(j+s)}(m) = lq^{j+s}\sum_{i=0}^{d}\binom{j+s}{i}\frac{d!}{(d-i)!}(n+h)^{d-i}Q^{i-j-s}f^{(j+s-i)}(n\alpha+r/q)$$

$$+ O((q/Q)^{j+s}N^d(1+(Q/N)^d)\epsilon_2\epsilon_4)$$

$$= ld!\,(q/Q)^{j+s}N^d\left[\sum_{i=0}^{d}\binom{j+s}{i}\frac{1}{(d-i)!}(Q/n)^i f^{(j+s-i)}(n\alpha+r/q)\right.$$

$$\left.+ O(\epsilon_2\epsilon_4(1+(Q/N)^d))\right].$$

Since $Q/N \gg \epsilon_0$, we can use Lemma 4.5 with $\epsilon = \delta = \epsilon_0$ to get that the number of (r,n)'s satisfying

$$\left|\sum_{i=0}^{d}\binom{j+s}{i}\frac{1}{(d-i)!}(Q/n)^i f^{(j+s-i)}(n\alpha+r/q)\right| \le a(\epsilon_0)\epsilon_0^{3d}(1+(Q/N)^d)/6^{d+1},$$

$$s=0,1$$

is $\ll Nq\epsilon_0 + N \ll Nq\epsilon_0$. For the remaining (r,n)'s, either

$$|g^{(j)}(m)| \gg \epsilon_0^{3d}\epsilon_2 lN^d(q/Q)^j$$

or

$$|g^{(j+1)}(m)| \gg \epsilon_0^{3d}\epsilon_2 lN^d(q/Q)^{j+1}.$$

Also, for any r and n, $|g^{(j)}(m)| \ll lN^d(q/Q)^j(1+(Q/N)^d) \ll \epsilon_0^J$ and $|g^{(j+1)}(m)| \ll \epsilon_0^J q/Q$. Applying Lemma 4.1 with $j=d+u$ or $j=d+u+1$, we obtain by (4.10) that

$$S \ll N\epsilon_0 + \frac{1}{H}\,[Nq\epsilon_0 H/q$$

$$+ Nq\max\{(H/q)\epsilon_0^{J/(J-2)} + 1 + (H/q)^{1-2/J}$$

$$+ (H/q)(\epsilon_0^{3d}\epsilon_2 lN^d(q/Q)^j H^2/q^2)^{-2/J},$$

$$(H/q)(\epsilon_0^J q/Q)^{1/(2J-2)} + 1 + (H/q)^{1-1/J}$$

$$+ (H/q)(\epsilon_0^{3d}\epsilon_2 lN^d(q/Q)^{j+1}H^2/q^2)^{-1/J}\}]$$

$$\ll N\epsilon_0.$$

This completes the proof.

Proof of Theorem 4.2. Using a well-known result of Erdös-Turán (cf. [2, Th. 2.1]), we get for any positive integer L

$$D(N) \leq \frac{1}{L+1} + \frac{3}{N} \sum_{l=1}^{L} \frac{1}{l} \left| \sum_{n=1}^{N} e(lP(n)f(n\alpha)) \right|.$$

Taking $L = [1/\epsilon(N)]$ we obtain

$$D(N) \ll \epsilon(N) \log L + \frac{1}{N} \sum_{l=1}^{L} \frac{1}{l} \sum_{i=1}^{\log_2 L} \left| \sum_{N/2^i \leq n \leq 2N/2^i} e(lP(n)f(n\alpha)) \right|.$$

One can check that the proof of Theorem 4.1 goes through if we take l depending on N and $\epsilon_0 = \epsilon(N)$, as long as the inequalities (i) and (ii) in that proof are satisfied. Now it is readily verified that conditions (1) and (2) in our theorem actually imply those inequalities. Hence

$$D(N) \ll \epsilon(N) \log^2 \epsilon(N) + \frac{1}{N} \sum_{i=1}^{\log_2 L} \sum_{l=1}^{L} \frac{1}{l} N\epsilon(N) \ll \epsilon(N) \log^2 \epsilon(N).$$

This proves the theorem.

Remark 4.2. In the course of the proof of Theorem 4.1 no attempt has been made to choose $\epsilon_1, \epsilon_2, \ldots$ in an optimal way. To the contrary, we have been quite generous with our choices, and one can obviously improve the result of Theorem 4.2 by just being more economical. Also, employing, instead of the van der Corput method, the Bombieri–Iwanieć method for "small" d and the Vinogradov method for "large" d, one can further sharpen the discrepancy estimate.

Example 4.1. Let $f(x) = e^{-1/\|x\|^\theta}$ for $x \notin \mathbf{Z}$ and $f(x) = 0$ otherwise, $\theta > 0$ arbitrary. If α is an irrational such that $b(N) \gg N^\eta$ for some $\eta > 0$ (note that a sufficient condition for this is that $|\alpha - p/q| \gg q^{-1/\eta}$, namely that α is not a Liouville number), then $D(N) \ll \log^2 \log N/(\log N)^{1/\theta}$, where the implied constant depends on θ, η and P only. In fact, one routinely verifies that $a(y) \gg e^{-1/y^\eta}$ and, taking $\epsilon(N) = C(\log N)^{-1/\eta}$ with a sufficiently large C, all the conditions of Theorem 4.2 are satisfied and we get the above bound for $D(N)$. Moreover, by examining the proof of Theorem 4.2, one can see that in this case $D(N) \ll \log\log N/(\log N)^{1/\theta}$, since $\sum_{i=1}^{\log_2 L} (N/2^i)(\log(N/2^i))^{-1/\theta} \ll N(\log N)^{-1/\theta}$. This result is actually pretty sharp. Indeed, if for some $1 \leq n \leq N$ we have $\|n\alpha\| < 1/((d+1)\log N)^{1/\theta}$, then $0 < n^d e^{-1/\|n\alpha\|^\theta} < 1/N$, whence the number of solutions $n \in [1,N]$ of $|P(n)e^{-1/\|n\alpha\|^\theta}| < 1/N$ is $\gg N/(\log N)^{1/\theta}$, which gives $D(N) \gg 1/(\log N)^{1/\theta}$.

Example 4.2. Let $P(x) = x$. Suppose that $T = 1, f \in C^3$ and f' and f'' have only finitely many zeros, each of which is simple. Let α be an irrational such that $b(N) \gg N^{5/8-\epsilon}$. In other words, for every sufficiently large M there exists a positive integer q such that $\alpha = p/q + \theta/qM$ with $|\theta| \leq 1$, $M^{1-\epsilon} \ll q \leq M$ (note that this includes all badly approximable numbers and also all α's for which $(Q_k^{1/k})_{k=1}^{\infty}$ is a convergent sequence, where Q_k is the denominator of the k-th convergent of α; the latter family contains a.e. real number). Obviously, $a(y) \gg y$. Applying Theorem 4.2 with $\epsilon(N) \sim N^{-1/280}$ we get $D(N) \ll N^{-1/280+\epsilon}$. If, instead of using Theorem 4.2 directly, we just employ the method utilized there with a more careful choice of parameters, we obtain $D(N) \ll \log N/N^{1/11}$. Moreover, the condition on α can be relaxed to $b(N) \gg N^{15/28}$. For simplicity, we shall prove it for $\alpha = p/q + \theta/qN^{7/11} = p/q + 1/Q$ with $Q \ll N^{29/22}$ and $N^{5/11} \ll q \ll N^{7/11}$. (The other case can be dealt with using Lemma 4.2 in exactly the same way as in Case I in the proof of Theorem 4.1.) As in Theorem 4.2 we write

$$D(N) \ll \frac{1}{L} + \frac{1}{N}\sum_{l=1}^{L}\frac{1}{l}\left|\sum_{n} e(lnf(n\alpha))\right| \ll \frac{1}{L} + \frac{1}{N}\sum_{l=1}^{L}\frac{1}{l}\sum_{k}\left|\sum_{m} e(g(m))\right|,$$

where $g(m) = lnf(r/q + n/Q)$. Here

$$g'''(m) = l((3q^3/Q^2)f''(r/q) + O(Nq^3/Q^3)).$$

The number of r's with $\rho(r/q,F) \ll N^{-1/11}$ is $\ll qN^{-1/11}$, while for the other r's we have $g'''(m) \sim l(q^3/Q^2)f''(r/q)$. Using Lemma 4.1 we obtain

$$\begin{aligned}
D(N) &\ll \frac{1}{L} + N^{-1/11} \\
&\quad + \frac{1}{N}\textstyle\sum_{l=1}^{L}\frac{1}{l}\sum_{k}^{*}\Big((N/q)\sqrt[6]{(lq^3/Q^2)} + 1 + (N/Q)^{3/4} \\
&\qquad + \sqrt{Q^2N/(lq^4f''(r/q))}\Big) \\
&\ll N^{-1/11} + \frac{1}{L} + \sqrt[6]{Lq^3/Q^2} + \sqrt[4]{q/N}\log N + \sqrt[4]{Q^2/N^3} \\
&\ll N^{-1/11}\log N + \sqrt[7]{Q^3/q^2} \\
&\ll \log N/N^{1/11}.
\end{aligned}$$

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